Acta Sci. Math., 50 (1986), 359—363

On measurable functions with a finite number of negative squares

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To Professor Heinz Langer on the occassion of his 50th birthday

1. Introduction

The complex-valued function f defined on the interval $(-2a, 2a)$, $0<a\leqq\infty$, is called Hermitian if $f(-x)=\overline{f(x)}$ for every $x\in(-2a, 2a)$. Let $\varkappa$ be a nonnegative integer. The Hermitian function f is said to have $\varkappa$ negative squares if the matrix

(1) $$(f(x_i-x_j))_{i,j=1}^n$$

has at most $\varkappa$ negative eigenvalues for any choice of n and $x_1, \ldots, x_n\in(-a, a)$, and for some choice of n and $x_1, \ldots, x_n$ the matrix (1) has exactly $\varkappa$ negative eigenvalues. Denote by $\mathfrak{P}^c_{\varkappa;a}$ ($\mathfrak{P}^m_{\varkappa;a}$) the class of all Hermitian functions defined on $(-2a, 2a)$ which are continuous (measurable, respectively) and which have $\varkappa$ negative squares.

In [1] the question was raised if $f\in\mathfrak{P}^m_{\varkappa;a}$ implies that f is locally bounded on $(-2a, 2a)$. The aim of this note is to answer this question in the affirmative (Theorem 1). Therefore in [1] in the definition of $\mathfrak{P}^m_{\varkappa;a}$ the condition of local boundednes of f can be dropped. We mention that an arbitrary positive definite function f (that is $\varkappa=0$) is bounded. If, however, $\varkappa>0$, then there exist nonmeasurable functions with $\varkappa$ negative sguares which are unbounded on each subinterval of $(-2a, 2a)$, see [1].

As a consequence of Theorem 1, of the decomposition result in [1] and of [3, Theorem 2] we show that an arbitrary function $f\in\mathfrak{P}^m_{\varkappa;a}$ has at least one continuation in $\mathfrak{P}^m_{\varkappa;\infty}$ (Theorem 3). An extension of Theorem 1 to locally compact groups is given in Section 3. A survey and a bibliography on functions with a finite number of negative squares can be found in [4].

Received April 28, 1984.

2. The main result

Theorem 1. *If $f\in\mathfrak{P}^m_{\varkappa;a}$ then f is locally bounded on* $(-2a, 2a)$.

Proof. We show that f is bounded on every interval $I_\delta=[-2a+\delta, 2a-\delta]$, $0<\delta<a$. For $c>0$ the function $f+c$ has at most $\varkappa$ negative squares. Therefore we can suppose that $f(0)>0$. If $K>0$, define

$$S_K f=\{t\in I\colon |f(t)|<K\}.$$

The sets $S_K f$ are measurable and we have

$$\lim_{K\to\infty}\lambda(S_K f)=4a$$

where λ denotes the Lebesgue measure. If f is not bounded on I_δ then there exists a sequence $\{t_n\}_1^\infty\subset I_\delta$ with $|f(t_n)|\to\infty$. Let $K>0$ be such that

$$\lambda(S_K f)>4a-\delta/\varkappa^3 \quad\text{and}\quad 0\in S_K f. \tag{2}$$

We show that for every $n=1, 2, \ldots$ there exist elements $x_1^{(n)}, \ldots, x_{\varkappa+1}^{(n)}$ with the following properties:

$$x_i^{(n)}\in[-\delta/2, \delta/2]=J_\delta \quad\text{for}\quad i=1, \ldots, \varkappa+1 \quad\text{and}\quad x_1^{(n)}=0, \tag{3}$$

$$x_i^{(n)}-x_j^{(n)}\in S_K f \quad\text{for}\quad i,j=1, \ldots, \varkappa+1, \tag{4}$$

$$x_i^{(n)}-x_j^{(n)}+t_n\in S_K f \quad\text{for}\quad i,j=1, \ldots, \varkappa+1, \quad i\neq j. \tag{5}$$

Let $x_1^{(n)}=0$, and suppose that $x_1^{(n)}, \ldots, x_l^{(n)}$, $1\leqq l<\varkappa+1$, have been found such that (3), (4) and (5) hold with $k+1$ replaced by l. For each $i=1, \ldots, l$ we define

$$M_i=(S_K f+x_i^{(n)})\cap J_\delta,\quad N_i=(S_K f+x_i^{(n)}-t_n)\cap J_\delta,\quad Q_i=(S_K f+x_i^{(n)}+t_n)\cap J_\delta.$$

From $|x_i^{(n)}|<\delta/2$, $|x_i^{(n)}\pm t_n|<2a-\delta/2$ and (2) it follows that

$$(\bigcap_{i,j,m=1}^{l}\{(J_\delta\setminus M_i)\cup(J_\delta\setminus N_j)\cup(J_\delta\setminus Q_m)\})<l^3\delta/\varkappa^3\leqq\delta=\lambda(J_\delta)$$

and so

$$\lambda(\bigcap_{i,j,m=1}^{l}(M_i\cap N_j\cap Q_m))>0.$$

Let $x_{l+1}^{(n)}$ be an arbitrary element of the set $\bigcap_{i,j,m=1}^{l}(M_i\cap N_j\cap Q_m)$. Then (3), (4) and (5) hold with $\varkappa+1$ replaced by $l+1$. Therefore, for every $n=1, 2, \ldots$ we can choose $x_1^{(n)}, \ldots, x_{\varkappa+1}^{(n)}$ so that (3), (4) and (5) are satisfied. Now let $z_1^{(n)}, \ldots, z_{2\varkappa+2}^{(n)}$ be defined by

$$z_1^{(n)}=x_1^{(n)},\quad z_2^{(n)}=x_1^{(n)}+t_n,\quad z_3^{(n)}=x_2^{(n)},$$

$$z_4^{(n)}=x_2^{(n)}+t_n, \ldots, z_{2\varkappa+1}^{(n)}=x_{\varkappa+1}^{(n)},\quad z_{2\varkappa+2}^{(n)}=x_{\varkappa+1}^{(n)}+t_n.$$

We consider the matrix

$$A^{(n)} = (a_{ij}^{(n)})_{i,j=1}^{2\varkappa+2} = \big(f(z_i^{(n)} - z_j^{(n)})\big)_{i,j=1}^{2\varkappa+2}.$$

The relations (3), (4) and (5) imply $|a_{2i-1,2i}^{(n)}| = |a_{2i,2i-1}^{(n)}| = |f(t_n)|$, $i=1, \dots, \varkappa+1$ and $|a_{ij}^{(n)}| < K$ for the other entries of $A^{(n)}$. Using these facts it is easy to see that by setting $D_r^{(n)} = \det (a_{ij}^{(n)})_{i,j=1}^r$, $r=1, \dots, 2\varkappa+2$, we have

$$\lim_{n\to\infty} D_{2r}^{(n)}/|f(t_n)|^{2r} = (-1)^r, \quad r = 1, \dots, \varkappa+1,$$

$$\lim_{n\to\infty} D_{2r+1}^{(n)}/|f(t_n)|^{2r} = (-1)^r, \quad r = 1, \dots, \varkappa,$$

$$D_1^{(n)} > 0.$$

It follows that for n sufficiently large, the signs in the sequence

$$1, D_1^{(n)}, D_2^{(n)}, \dots, D_{2\varkappa+2}^{(n)}$$

change exactly $\varkappa+1$ times. Consequently, by Frobenius's rule the matrix $A^{(n)}$ has $\varkappa+1$ negative eigenvalues. This is a contradiction to the assumption that f has exactly $\varkappa$ negative squares. The theorem is proved.

Combining Theorem 1 and the decomposition theorem in [1] we have the following

Theorem 2. *Every function $f \in \mathfrak{P}_{\varkappa;a}^m$ admits a unique decomposition*

$$f(t) = f_c(t) + f_s(t) \quad (-2a < t < 2a) \tag{7}$$

such that $f_c \in \mathfrak{P}_{\varkappa;a}^{c}$, $f_s \in \mathfrak{P}_{0;a}^m$ and $f_s(t)=0$ a.e. on $(-2a, 2a)$.

Corollary 1. *If a is finite then $f \in \mathfrak{P}_{\varkappa;a}^m$ is bounded on $(-2a, 2a)$.*

Indeed, in the decomposition (7) the function f_c is bounded on $(-2a, 2a)$ according to [2] and f_s is bounded as it is a positive definite function.

Theorem 3. *Let $f \in \mathfrak{P}_{\varkappa;a}^m$ where $0 < a < \infty$ and $\varkappa$ is a nonnegative integer. Then there exists a function $\tilde{f} \in \mathfrak{P}_{\varkappa;\infty}^m$ such that*

$$f(t) = \tilde{f}(t) \quad (-2a < t < 2a).$$

Proof. By Theorem 2 the function f admits a decomposition

$$f(t) = f_c(t) + f_s(t) \quad (-2a < t < 2a)$$

such that $f_c \in \mathfrak{P}_{\varkappa;a}^c$, $f_s \in \mathfrak{P}_{0;a}^m$ and $f_s(t)=0$ a.e. on $(-2a, 2a)$. In [2] it was shown that f_c has an extension $\tilde{f}_c \in \mathfrak{P}_{\varkappa;\infty}^c$, and by Theorem 2 in [3] f_s has an extension $\tilde{f}_s \in \mathfrak{P}_{\varkappa;\infty}^m$. If we set $\tilde{f} = \tilde{f}_c + \tilde{f}_s$ we have $\tilde{f} \in \mathfrak{P}_{\varkappa;\infty}^m$ and $f(t) = \tilde{f}(t)$ $(-2a < t < 2a)$.

3. Some remarks

1. Let $\varkappa$ again be a nonnegative integer. The definition of a function with $\varkappa$ negative squares can be formulated on a group G as follows. Suppose that f is a complex-valued function defined on a symmetric set $V\subset G$ which contains the unit of G and has the property that $f(g^{-1})=\overline{f(g)}$ for every $g\in V$. The function f is said to have $\varkappa$ negative squares if the matrix

$$(8) \qquad (f(g_i g_j^{-1}))_{i,j=1}^{n}$$

has at most $\varkappa$ negative eigenvalues for any choice of n and $g_1, \ldots, g_n \in V$ for which $g_i g_j^{-1} \in V$ $(i,j=1, \ldots, n)$, and if for some choice of n and $g_1, \ldots, g_n$ the matrix (8) has exactly $\varkappa$ negative eigenvalues. We denote by $\mathfrak{P}_{\varkappa;V}$ the set of functions on V which have $\varkappa$ negative squares. If G is a locally compact group with Haar measure λ and V is λ-measurable we set

$$\mathfrak{P}^m_{\varkappa;V} = \{f\in \mathfrak{P}_{\varkappa;V} \colon f \text{ is } \lambda\text{-measurable on } V\}.$$

It is not hard to see that the arguments in the proof of Theorem 1 can be used in order to show the following result.

Theorem 1′. *Let G be a locally compact group and $f\in \mathfrak{P}^m_{\varkappa;V}$ where V is an open symmetric subset of G. Then f is locally bounded, that is, f is bounded on every compact set $K\subset V$.*

2. It follows immediately from Theorem 2 that a function $f\in \mathfrak{P}^m_{\varkappa;a}$ with $\varkappa \geqq 1$ cannot vanish almost everywhere on $(-2a, 2a)$. This fact remains valid for functions on a locally compact group. In order to see this we need the following.

Lemma 1. *Let G be an arbitrary group and $f\in \mathfrak{P}_{\varkappa;V}$. If $\varkappa\geqq 1$ then V can be covered with finitely many translates of the support of f.*

Proof. Let $g_1, \ldots, g_n \in V$ be such that the matrix $A=(f(g_i g_j^{-1}))_{i,j=1}^{n}$ has $\varkappa$ negative eigenvalues. Suppose that V cannot be covered with finitely many translates of $\operatorname{Supp} f=\{g\in V\colon f(g)\neq 0\}$. Then there exists a $g\in V$ for which $g\notin g_i (\operatorname{Supp} f) g_j^{-1}$, $i,j=1, \ldots, n$. Let $t_1, \ldots, t_{2n}$ be defined by

$$t_1 = g_1, \ldots, t_n = g_n, \quad t_{n+1} = g_1 g, \ldots, t_{2n} = g_n g.$$

Then we have

$$B = (f(t_i t_j^{-1}))_{i,j=1}^{2n} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Thus B has $2\varkappa$ negative eigenvalues, a contradiction.

Corollary 2. *Let G be a locally compact group and let* $f\in\mathfrak{P}^{m}_{\varkappa;V}$ *where* $\lambda(V)>0$. *If* $\varkappa\geqq 1$ *then* f *cannot vanish almost everywhere on* V.

The author is indebted to Professor Heinz Langer for suggesting the problem discussed in this paper and for several helpful discussions.

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